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Some New Properties for k -Fibonacci and k -Lucas Numbers using Matrix Methods

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Abstract

In this paper, some new sums for k -Fibonacci and k -Lucas numbers are derived and proved by using matrices $S = \begin{bmatrix} \frac{k}{2} & \frac{k^2+4}{2} \\ \frac{1}{2} & \frac{k}{2} \end{bmatrix}$ and $R = \begin{bmatrix} 0 & k^2+4 \\ 1 & 0 \end{bmatrix}$. The sums we derived are not encountered in the k -Fibonacci and k -Lucas numbers literature.

Keywords: k -Fibonacci sequence; k -Lucas sequence; Recurrence relation; Matrix algebra.

Mathematics Subject Classification (2010): 11B39, 11B37, 11B99.

1 Introduction

Many authors defined generalized Fibonacci sequences by varying initial conditions and recurrence relation [1 – 14]. This paper represents an interesting investigation about some special relations between matrices and k -Fibonacci numbers, k -Lucas numbers. This investigation is valuable to obtain new k -Fibonacci, k -Lucas identities by different methods. This paper contributes to k -Fibonacci, k -Lucas numbers literature, and encourage many researchers to investigate the properties of such number sequences.

2 Some Preliminary Results

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 2.1 The k -Fibonacci sequence $\{F_{k,n}\}_{n \in \mathbb{N}}$ is defined as, $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, with $F_{k,0} = 0, F_{k,1} = 1$, for $n \geq 1$

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Definition 2.2 The k - Lucas sequence $\{L_{k,n}\}_{n \in \mathbb{N}}$ is defined as, $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$, with $L_{k,0} = 2, L_{k,1} = k$, for $n \geq 1$

Characteristic equation of the initial recurrence relation is,

$$r^2 - kr - 1 = 0, \quad (1)$$

and characteristic roots are

$$r_1 = \frac{k + \sqrt{k^2 + 4}}{2}$$

and

$$r_2 = \frac{k - \sqrt{k^2 + 4}}{2},$$

Characteristic roots verify the properties,

$$r_1 - r_2 = \sqrt{k^2 + 4} = \sqrt{\Delta}, \quad (2)$$

$$r_1 + r_2 = k, \quad (3)$$

$$r_1 \cdot r_2 = -1. \quad (4)$$

Binet forms for $F_{k,n}$ and $L_{k,n}$ are

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad (5)$$

and

$$L_{k,n} = r_1^n + r_2^n \quad (6)$$

From the definition of the k -Fibonacci numbers, one may deduce the value of any k -Fibonacci number by simple substitution on the corresponding $F_{k,n}$. For example, the seventh element of the 4-Fibonacci sequence, $F_{4,7}$ is 5473. By doing $k = 1; 2; 3; \dots$ the respective k -Fibonacci sequences are obtained. Sequence $\{F_{1,n}\}$ is the classical Fibonacci sequence and $\{F_{2,n}\}$ is the Pell sequence. It is worthy to be noted that only the first 10 k -Fibonacci sequences are referenced in The On-Line Encyclopedia of Integer Sequences [11] with the numbers given in Table 1. For k even with $12 \leq k \leq 62$ sequences $\{F_{k,n}\}$ are referenced without the first term $F_{k,0} = 0$ in [11].

The first 11 k -Fibonacci sequences as numbered in The On-Line Encyclopedia of Integer Sequences [10]:

$F_{1,n}$	A000045
$F_{2,n}$	A000129
$F_{3,n}$	A006190
$F_{4,n}$	A001076
$F_{5,n}$	A052918
$F_{6,n}$	A005668
$F_{7,n}$	A054413
$F_{8,n}$	A041025
$F_{9,n}$	A099371
$F_{10,n}$	A041041
$F_{11,n}$	A049666

The most commonly used matrix in relation to the recurrence relation(1) is

$$M = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix} \quad (7)$$

which for $k = 1$ reduces to the ordinary Q -matrix [4],[8]. In this paper we define more general matrices $M_k(n, m), T_k(n), S_k(n, m)$. We use these matrices to develop various summation identities involving terms from the sequences $F_{k,n}$ and $L_{k,n}$.

Several identities for $F_{k,n}$ and $L_{k,n}$ are proved using Binet forms in [1],[3] and [5].Some of these are listed below.

$$F_{k,n+1} + F_{k,n-1} = L_{k,n} \quad (8)$$

$$F_{k,n+1} + L_{k,n-1} = \Delta F_{k,n} \quad (9)$$

$$F_{k,2n} - 2(-1)^n = \Delta F_{k,n}^2 \quad (10)$$

$$F_{k,m+n} - (-1)^m L_{k,n-m} = F_{k,m} L_{k,n} \quad (11)$$

$$L_{k,m+n} - (-1)^m L_{k,n-m} = \Delta F_{k,m} F_{k,n} \quad (12)$$

$$F_{k,m+n} F_{k,n-m} - F_{k,n}^2 = (-1)^{n-m+1} F_{k,m}^2 \quad (13)$$

$$L_{k,m+n} L_{k,n-m} - L_{k,n}^2 = (-1)^{n-m} F_{k,m}^2 \quad (14)$$

$$F_{k,m+n} F_{k,r+m} - (-1)^m F_{k,n} F_{k,r} = F_{k,m} F_{k,n+r+m} \quad (15)$$

$$L_{k,mn} L_{k,n} + \Delta F_{k,mn} F_{k,n} = 2L_{k,(m+1)n} \quad (16)$$

$$F_{k,mn} L_{k,n} + L_{k,mn} F_{k,n} = 2F_{k,(m+1)n} \quad (17)$$

$$L_{k,m+n}^2 + (-1)^{m-1} L_{k,n}^2 = \Delta F_{k,2n+m} F_{k,m} \quad (18)$$

$$L_{k,m+n} L_{k,n} + (-1)^{m+1} L_{k,n-m} L_{k,n} = \Delta F_{k,2n} F_{k,m} \quad (19)$$

$$F_{k,m+2rn} F_{k,2n+m} + (-1)^{m+1} F_{k,2rn} F_{k,2n} = F_{k,2(r+1)n+m} F_{k,m} \quad (20)$$

In [14]matrix M is generalized using PMI

$$M^n = \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix} \quad (21)$$

where n is an integer.

Lemma 2.3 *If X is a square matrix with $X^2 = kX + I$, then $X^n = F_{k,n}X + F_{k,n-1}I$, for all $n \in \mathbb{Z}$*

Corollary 2.4 *Let, $M = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$, then $M^n = \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix}$*

Proof. Since,

$$\begin{aligned} M^2 &= kM + I \\ &= F_{k,n}M + F_{k,n-1}I \text{ (Using Lemma 2.1)} \\ &= \begin{bmatrix} kF_{k,n} & F_{k,n} \\ F_{k,n} & 0 \end{bmatrix} + \begin{bmatrix} F_{k,n-1} & 0 \\ 0 & F_{k,n-1} \end{bmatrix} \\ &= \begin{bmatrix} F_{k,n+1} & F_{k,n} \\ F_{k,n} & F_{k,n-1} \end{bmatrix}, \end{aligned}$$

for all $n \in \mathbb{Z}$

□

Corollary 2.5 *Let, $S = \begin{bmatrix} \frac{k}{2} & \frac{\Delta}{2} \\ \frac{k}{2} & \frac{\Delta}{2} \end{bmatrix}$, then $S^n = \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{(\Delta)F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}$, for every $n \in \mathbb{Z}$*

Lemma 2.6

$$L_{k,n}^2 - \Delta F_{k,n}^2 = 4(-1)^n$$

for all, $n \in \mathbb{Z}$

Proof. Since,

$$\det(S) = -1$$

$$\det(S^n) = [\det(S)]^n = (-1)^n$$

Moreover since,

$$S^n = \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{\Delta F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix}$$

We get,

$$\det(S^n) = \frac{L_{k,n}^2}{4} - \frac{\Delta F_{k,n}^2}{4}$$

Thus it follows that,

$$L_{k,n}^2 - \Delta F_{k,n}^2 = 4(-1)^n,$$

for all $n \in \mathbb{Z}$

□

Lemma 2.7

$$2L_{k,n+m} = L_{k,n}L_{k,m} + \Delta F_{k,n}F_{k,m}$$

and

$$2F_{k,n+m} = F_{k,n}L_{k,m} + L_{k,n}F_{k,m}$$

for all $n, m \in Z$

Proof. Since,

$$\begin{aligned} S^{n+m} &= S^n \cdot S^m \\ &= \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{\Delta F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{L_{k,m}}{2} & \frac{\Delta F_{k,m}}{2} \\ \frac{F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{L_{k,n}L_{k,m} + \Delta F_{k,n}F_{k,m}}{4} & \frac{\Delta[L_{k,n}F_{k,m} + F_{k,n}L_{k,m}]}{4} \\ \frac{L_{k,n}F_{k,m} + F_{k,n}L_{k,m}}{4} & \frac{L_{k,n}L_{k,m} + \Delta F_{k,n}F_{k,m}}{4} \end{bmatrix} \end{aligned}$$

But,

$$S^{n+m} = \begin{bmatrix} \frac{L_{k,n+m}}{2} & \frac{\Delta F_{k,n+m}}{2} \\ \frac{F_{k,n+m}}{2} & \frac{L_{k,n+m}}{2} \end{bmatrix}$$

Gives,

$$2L_{k,n+m} = L_{k,n}L_{k,m} + \Delta F_{k,n}F_{k,m}$$

and

$$2F_{k,n+m} = F_{k,n}L_{k,m} + L_{k,n}F_{k,m}$$

for all $n, m \in Z$

□

Lemma 2.8

$$2(-1)^m L_{k,n-m} = L_{k,n}L_{k,m} - \Delta F_{k,n}F_{k,m}$$

and

$$2(-1)^m F_{k,n-m} = F_{k,n}L_{k,m} - L_{k,n}F_{k,m}$$

for all $n, m \in Z$

Proof. Since,

$$\begin{aligned} S^{n-m} &= S^n \cdot S^{-m} \\ &= S^n \cdot [S^m]^{-1} \\ &= S^n \cdot (-1)^m \begin{bmatrix} \frac{L_{k,m}}{2} & \frac{-\Delta F_{k,m}}{2} \\ \frac{-F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix} \\ &= (-1)^m \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{\Delta F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{L_{k,m}}{2} & \frac{-\Delta F_{k,m}}{2} \\ \frac{-F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{L_{k,n}L_{k,m} - \Delta F_{k,n}F_{k,m}}{4} & \frac{\Delta[L_{k,n}F_{k,m} - F_{k,n}L_{k,m}]}{4} \\ \frac{L_{k,n}F_{k,m} - F_{k,n}L_{k,m}}{4} & \frac{L_{k,n}L_{k,m} - \Delta F_{k,n}F_{k,m}}{4} \end{bmatrix} \end{aligned}$$

But,

$$S^{n-m} = \begin{bmatrix} \frac{L_{k,n-m}}{2} & \frac{\Delta F_{k,n-m}}{2} \\ \frac{F_{k,n-m}}{2} & \frac{L_{k,n-m}}{2} \end{bmatrix}$$

Gives,

$$2(-1)^m L_{k,n-m} = L_{k,n} L_{k,m} - \Delta F_{k,n} F_{k,m}$$

and

$$2(-1)^m F_{k,n-m} = F_{k,n} L_{k,m} - L_{k,n} F_{k,m}$$

for all $n, m \in \mathbb{Z}$.

Lemma 2.9

$$(-1)^m L_{k,n-m} + L_{k,n+m} = L_{k,n} L_{k,m}$$

and

$$(-1)^m F_{k,n-m} + F_{k,n+m} = F_{k,n} L_{k,m}$$

for all $n, m \in \mathbb{Z}$

Proof. By definition of the matrix S^n , it can be seen that

$$S^{n+m} + (-1)^m S^{n-m} = \begin{bmatrix} \frac{L_{k,n+m} + (-1)^m L_{k,n-m}}{2} & \frac{\Delta[F_{k,n+m} + (-1)^m F_{k,n-m}]}{2} \\ \frac{F_{k,n+m} + (-1)^m F_{k,n-m}}{2} & \frac{L_{k,n+m} + (-1)^m L_{k,n-m}}{2} \end{bmatrix}$$

On the other hand,

$$\begin{aligned} S^{n+m} + (-1)^m S^{n-m} &= S^n S^m + (-1)^m S^n S^{-m} \\ &= S^n [S^m + (-1)^m S^{-m}] \\ &= \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{\Delta F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{L_{k,m}}{2} & \frac{\Delta F_{k,m}}{2} \\ \frac{F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix} + (-1)^m \begin{bmatrix} \frac{L_{k,m}}{2} & \frac{-\Delta F_{k,m}}{2} \\ \frac{-F_{k,m}}{2} & \frac{L_{k,m}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{L_{k,n}}{2} & \frac{\Delta F_{k,n}}{2} \\ \frac{F_{k,n}}{2} & \frac{L_{k,n}}{2} \end{bmatrix} \cdot \begin{bmatrix} L_{k,m} & 0 \\ 0 & L_{k,m} \end{bmatrix} \\ &= \begin{bmatrix} \frac{L_{k,m} L_{k,n}}{2} & \frac{\Delta F_{k,n} L_{k,m}}{2} \\ \frac{F_{k,n} L_{k,m}}{2} & \frac{L_{k,m} L_{k,n}}{2} \end{bmatrix} \end{aligned}$$

Gives,

$$(-1)^m L_{k,n-m} + L_{k,n+m} = L_{k,n} L_{k,m}$$

and

$$(-1)^m F_{k,n-m} + F_{k,n+m} = F_{k,n} L_{k,m}$$

for all $n, m \in \mathbb{Z}$

□

Lemma 2.10

$$8F_{k,x+y+z} = L_{k,x}L_{k,y}F_{k,z} + F_{k,x}L_{k,y}L_{k,z} + L_{k,x}F_{k,y}L_{k,z} + \Delta F_{k,x}F_{k,y}F_{k,z}$$

and

$$8L_{k,x+y+z} = L_{k,x}L_{k,y}L_{k,z} + \Delta[L_{k,x}F_{k,y}F_{k,z} + F_{k,x}L_{k,y}F_{k,z} + F_{k,x}F_{k,y}L_{k,z}]$$

for all $x, y, z \in Z$

Proof. By definition of the matrix S^n , it can be seen that

$$S^{x+y+z} = \begin{bmatrix} \frac{L_{k,x+y+z}}{2} & \frac{\Delta F_{k,x+y+z}}{2} \\ \frac{F_{k,x+y+z}}{2} & \frac{L_{k,x+y+z}}{2} \end{bmatrix}$$

On the other hand,

$$\begin{aligned} S^{x+y+z} &= S^{x+y} S^z \\ &= \begin{bmatrix} \frac{L_{k,x+y}}{2} & \frac{\Delta F_{k,x+y}}{2} \\ \frac{F_{k,x+y}}{2} & \frac{L_{k,x+y}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{L_{k,z}}{2} & \frac{\Delta F_{k,z}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{L_{k,x+y}L_{k,z} + \Delta F_{k,x+y}F_{k,z}}{4} & \frac{\Delta[L_{k,x+y}F_{k,z} + F_{k,x+y}L_{k,z}]}{4} \\ \frac{L_{k,z}F_{k,x+y} + F_{k,z}L_{k,x+y}}{4} & \frac{L_{k,x+y}L_{k,z} + \Delta F_{k,x+y}F_{k,z}}{4} \end{bmatrix} \end{aligned}$$

Using,

$$2L_{k,x+y} = L_{k,x}L_{k,y} + \Delta F_{k,x}F_{k,y}$$

$$2F_{k,x+y} = L_{k,y}F_{k,x} + \Delta F_{k,y}L_{k,x}$$

Gives,

$$8F_{k,x+y+z} = L_{k,x}L_{k,y}F_{k,z} + F_{k,x}L_{k,y}L_{k,z} + L_{k,x}F_{k,y}L_{k,z} + \Delta F_{k,x}F_{k,y}F_{k,z}$$

and

$$8L_{k,x+y+z} = L_{k,x}L_{k,y}L_{k,z} + \Delta[L_{k,x}F_{k,y}F_{k,z} + F_{k,x}L_{k,y}F_{k,z} + F_{k,x}F_{k,y}L_{k,z}]$$

for all $x, y, z \in Z$

□

Theorem 2.11

$$L_{k,x+y}^2 - \Delta(-1)^{x+y+1}F_{k,z-x}L_{k,x+y}F_{k,y+z} - \Delta(-1)^{x+z}F_{k,y+z}^2 = (-1)^{y+z}L_{k,z-x}^2$$

for all $x, y, z \in Z$

Proof. Consider matrix multiplication given below.

$$\begin{bmatrix} \frac{L_{k,x}}{2} & \frac{\Delta F_{k,x}}{2} \\ \frac{F_{k,x}}{2} & \frac{L_{k,x}}{2} \end{bmatrix} \cdot \begin{bmatrix} L_{k,y} \\ F_{k,y} \end{bmatrix} = \begin{bmatrix} L_{k,x+y} \\ F_{k,y+z} \end{bmatrix}$$

Now,

$$\begin{aligned}
\det. \begin{bmatrix} \frac{L_{k,x}}{2} & \frac{\Delta F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix} &= \frac{L_{k,x}L_{k,z} - \Delta F_{k,x}F_{k,z}}{4} \\
&= \frac{(-1)^x L_{k,z-x}}{2} \\
&= Q \\
&\neq 0
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
\begin{bmatrix} L_{k,y} \\ F_{k,y} \end{bmatrix} &= \begin{bmatrix} \frac{L_{k,x}}{2} & \frac{\Delta F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} L_{k,x+y} \\ F_{k,y+z} \end{bmatrix} \\
&= \frac{1}{Q} \begin{bmatrix} \frac{L_{k,z}}{2} & \frac{-\Delta F_{k,x}}{2} \\ \frac{-F_{k,z}}{2} & \frac{L_{k,x}}{2} \end{bmatrix} \cdot \begin{bmatrix} L_{k,x+y} \\ F_{k,y+z} \end{bmatrix}
\end{aligned}$$

Gives,

$$L_{k,y} = \frac{(-1)^x [L_{k,z}L_{k,x+y} - \Delta F_{k,x}F_{k,y+z}]}{L_{k,z-x}}$$

and

$$F_{k,y} = \frac{(-1)^x [L_{k,x}F_{k,z+y} - F_{k,z}L_{k,y+x}]}{L_{k,z-x}}$$

Since,

$$L_{k,y}^2 - \Delta F_{k,y}^2 = 4(-1)^y$$

We get,

$$[L_{k,z}L_{k,x+y} - \Delta F_{k,x}F_{k,y+z}]^2 - (\Delta)^2 [L_{k,x}F_{k,z+y} - F_{k,z}L_{k,y+x}]^2 = 4(-1)^y L_{k,z-x}^2$$

Using Lemma 2.4 and Lemma 2.6,

$$\begin{aligned}
&(L_{k,z}^2 L_{k,x+y}^2 - 2\Delta L_{k,z} F_{k,x+y} F_{k,y+z} + \Delta^2 F_{k,x}^2 F_{k,y+z}^2) \\
&- \Delta (L_{k,x}^2 F_{k,y+z}^2 - 2L_{k,x} F_{k,z} F_{k,y+z} L_{k,x+y} + F_{k,z}^2 L_{k,x+y}^2) \\
&= 4(-1)^y L_{k,z-x}^2
\end{aligned}$$

Gives,

$$L_{k,x+y}^2 - \Delta(-1)^{x+y+1} F_{k,z-x} L_{k,x+y} F_{k,y+z} - \Delta(-1)^{x+z} F_{k,y+z}^2 = (-1)^{y+z} L_{k,z-x}^2$$

for all $x, y, z \in Z$

□

Theorem 2.12

$$L_{k,x+y}^2 - (-1)^{x+z} L_{k,z-x} L_{k,x+y} L_{k,y+z} + (-1)^{x+z} L_{k,y+z}^2 = (-1)^{y+z+1} \Delta F_{k,z-x}^2$$

for all $x, y, z \in Z, x \neq z$

Proof. Consider matrix multiplication,

$$\begin{bmatrix} \frac{L_{k,x}}{2} & \frac{\Delta F_{k,x}}{2} \\ \frac{L_{k,z}}{2} & \frac{\Delta F_{k,z}}{2} \end{bmatrix} \cdot \begin{bmatrix} L_{k,y} \\ F_{k,y} \end{bmatrix} = \begin{bmatrix} L_{k,x+y} \\ L_{k,y+z} \end{bmatrix}$$

Now,

$$\begin{aligned} \det. \begin{bmatrix} \frac{L_{k,x}}{2} & \frac{\Delta F_{k,x}}{2} \\ \frac{L_{k,z}}{2} & \frac{\Delta F_{k,z}}{2} \end{bmatrix} &= \frac{\Delta(-1)^x F_{k,z-x}}{2} \\ &= P \\ &\neq 0, (if x \neq z) \end{aligned}$$

Therefore for $x \neq z$, we can write

$$\begin{aligned} \begin{bmatrix} L_{k,y} \\ F_{k,y} \end{bmatrix} &= \begin{bmatrix} \frac{L_{k,x}}{2} & \frac{\Delta F_{k,x}}{2} \\ \frac{L_{k,z}}{2} & \frac{\Delta F_{k,z}}{2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} L_{k,x+y} \\ L_{k,y+z} \end{bmatrix} \\ &= \frac{1}{P} \begin{bmatrix} \frac{(k^2+4)F_{k,z}}{2} & \frac{-\Delta F_{k,x}}{2} \\ \frac{-L_{k,z}}{2} & \frac{L_{k,x}}{2} \end{bmatrix} \cdot \begin{bmatrix} L_{k,x+y} \\ L_{k,y+z} \end{bmatrix} \end{aligned}$$

Gives,

$$L_{k,y} = \frac{(-1)^x [F_{k,z} L_{k,x+y} - F_{k,x} L_{k,y+z}]}{F_{k,z-x}}$$

and

$$F_{k,y} = \frac{(-1)^x [L_{k,x} L_{k,z+y} - L_{k,z} L_{k,y+x}]}{\Delta F_{k,z-x}}$$

Since,

$$L_{k,y}^2 - \Delta F_{k,y}^2 = 4(-1)^y$$

We get,

$$\Delta [F_{k,z} L_{k,x+y} - F_{k,x} L_{k,y+z}]^2 - [L_{k,x} L_{k,z+y} - L_{k,z} L_{k,y+x}]^2 = 4\Delta(-1)^y F_{k,z-x}^2$$

Using Lemma 2.4 and Lemma 2.6, We obtain

$$L_{k,x+y}^2 - (-1)^{x+z} L_{k,z-x} L_{k,x+y} L_{k,y+z} + (-1)^{x+z} L_{k,y+z}^2 = (-1)^{y+z+1} \Delta F_{k,z-x}^2$$

for all $x, y, z \in Z, x \neq z$

□

Theorem 2.13

$$F_{k,x+y}^2 - L_{k,x-z}F_{k,x+y}F_{k,y+z} + (-1)^{x+z}F_{k,y+z}^2 = (-1)^{y+z}F_{k,z-x}^2$$

for all $x, y, z \in Z, x \neq z$

Proof. Consider matrix multiplication,

$$\begin{bmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix} \cdot \begin{bmatrix} L_{k,y} \\ F_{k,y} \end{bmatrix} = \begin{bmatrix} F_{k,x+y} \\ F_{k,y+z} \end{bmatrix}$$

Now,

$$\begin{aligned} \det. \begin{bmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix} &= \frac{(-1)^z F_{k,x-z}}{2} \\ &= T \\ &\neq 0, (if x \neq z) \end{aligned}$$

Therefore for $x \neq z$, we get,

$$\begin{aligned} \begin{bmatrix} L_{k,y} \\ F_{k,y} \end{bmatrix} &= \begin{bmatrix} \frac{F_{k,x}}{2} & \frac{F_{k,x}}{2} \\ \frac{F_{k,z}}{2} & \frac{L_{k,z}}{2} \end{bmatrix}^{-1} \cdot \begin{bmatrix} F_{k,x+y} \\ F_{k,y+z} \end{bmatrix} \\ &= \frac{1}{T} \begin{bmatrix} \frac{L_{k,z}}{2} & \frac{-L_{k,x}}{2} \\ \frac{-F_{k,z}}{2} & \frac{F_{k,x}}{2} \end{bmatrix} \cdot \begin{bmatrix} F_{k,x+y} \\ F_{k,y+z} \end{bmatrix} \end{aligned}$$

Gives,

$$L_{k,y} = \frac{(-1)^z [L_{k,z}F_{k,x+y} - L_{k,x}F_{k,y+z}]}{F_{k,x-z}}$$

and

$$F_{k,y} = \frac{(-1)^z [F_{k,x}F_{k,z+y} - F_{k,z}F_{k,y+x}]}{F_{k,x-z}}$$

Now consider,,

$$[L_{k,z}F_{k,x+y} - L_{k,x}F_{k,y+z}]^2 - \Delta[F_{k,x}F_{k,z+y} - F_{k,z}F_{k,y+x}]^2 = 4(-1)^y F_{k,x-z}^2$$

Using Lemma 2.4 and Lemma 2.6, We get

$$F_{k,x+y}^2 - L_{k,x-z}F_{k,x+y}F_{k,y+z} + (-1)^{x+z}F_{k,y+z}^2 = (-1)^{y+z}F_{k,z-x}^2$$

for all $x, y, z \in Z, x \neq z$

□

3 Main Results

Theorem 3.1 *Let $n \in N$ and $m, t \in Z$ with $m \neq 0$, Then*

$$\sum_{j=0}^{j=n} L_{k,mj+t} = \frac{L_{k,t} - L_{k,mn+m+t} + (-1)^m (L_{k,mn+t} - L_{k,t-m})}{1 + (-1)^m - L_{k,m}} \quad (21)$$

and

$$\sum_{j=0}^{j=n} F_{k,mj+t} = \frac{F_{k,t} - F_{k,mn+m+t} + (-1)^m (F_{k,mn+t} - F_{k,t-m})}{1 + (-1)^m - L_{k,m}} \quad (22)$$

Proof. It is known that,

$$I - (S^m)^{n+1} = (I - S^m) \sum_{j=0}^{j=n} (S^m)^j$$

If $\det(I - S^m) \neq 0$, then we can write

$$I - (S^m)^{-1} (I - (S^m)^{n+1}) S^t = \sum_{j=0}^{j=n} (S^{mj+t}) = \begin{bmatrix} \frac{1}{2} \sum_{j=0}^{j=n} (L_{k,mj+t}) & \frac{\Delta}{2} \sum_{j=0}^{j=n} (F_{k,mj+t}) \\ \frac{1}{2} \sum_{j=0}^{j=n} (F_{k,mj+t}) & \frac{1}{2} \sum_{j=0}^{j=n} (L_{k,mj+t}) \end{bmatrix} \quad (23)$$

From lemma (2.4)

$$\det(I - S^m) = (1 - \frac{L_{k,m}}{2})^2 - \Delta \frac{(F_{k,m})^2}{4} = 1 + (-1)^m - L_{k,m} \neq 0$$

For $m \neq 0$, If we take $D = 1 + (-1)^m - L_{k,m}$, then we get

$$\begin{aligned} (I - S^m)^{-1} &= \frac{1}{D} \begin{bmatrix} 1 - \frac{(L_{k,m})}{2} & \Delta \frac{(F_{k,m})}{2} \\ \frac{(F_{k,m})}{2} & 1 - \frac{(L_{k,m})}{2} \end{bmatrix} \\ &= \frac{1}{D} \left[\left(1 - \frac{(L_{k,m})}{2}\right) I + \frac{(F_{k,m})}{2} R \right] \end{aligned}$$

Thus it is seen that,

$$(I - S^m)^{-1} (S^t - S^{mn+m+t}) = \frac{1}{D} \left[\left(1 - \frac{(L_{k,m})}{2}\right) I + \frac{(F_{k,m})}{2} R \right] (S^t - S^{mn+m+t})$$

$$(I - S^m)^{-1} (S^t - S^{mn+m+t}) = \frac{1}{D} \left[\left(1 - \frac{(L_{k,m})}{2}\right) (S^t - S^{mn+m+t}) + \frac{(F_{k,m})}{2} R (S^t - S^{mn+m+t}) \right] \quad (24)$$

By Corollary (2.2) and (2.3) we get

$$R(S^t - S^{mn+m+t}) = \begin{bmatrix} \frac{\Delta (F_{k,t} - F_{k,mn+m+t})}{2} & \frac{\Delta (L_{k,t} - L_{k,mn+m+t})}{2} \\ \frac{(L_{k,t} - L_{k,mn+m+t})}{2} & \frac{\Delta (F_{k,t} - F_{k,mn+m+t})}{2} \end{bmatrix} \quad (25)$$

Using 24 in 23, we obtain

$$\begin{aligned} (I - S^m)^{-1}(S^t - S^{mn+m+t}) &= \frac{1}{D} \left(1 - \frac{L_{k,m}}{2}\right) \begin{bmatrix} \Delta \frac{(L_{k,t} - L_{k,mn+m+t})}{2} & \Delta \frac{(F_{k,t} - F_{k,mn+m+t})}{2} \\ \frac{(F_{k,t} - F_{k,mn+m+t})}{2} & \frac{(L_{k,t} - L_{k,mn+m+t})}{2} \end{bmatrix} \\ &+ \frac{F_{k,m}}{2D} \begin{bmatrix} \Delta \frac{(F_{k,t} - F_{k,mn+m+t})}{2} & \Delta \frac{(L_{k,t} - L_{k,mn+m+t})}{2} \\ \frac{(L_{k,t} - L_{k,mn+m+t})}{2} & \Delta \frac{(F_{k,t} - F_{k,mn+m+t})}{2} \end{bmatrix} \end{aligned}$$

Thus from (22) it follows that

$$\begin{aligned} \sum_{j=0}^{j=n} L_{k,mj+t} &= \frac{1}{D} \left[\left(1 - \frac{L_{k,m}}{2}\right)(L_{k,t} - L_{k,mn+m+t}) + \frac{F_{k,m}\Delta}{2}(F_{k,t} - F_{k,mn+m+t}) \right] \\ &= \frac{1}{D} \left[L_{k,t} - L_{k,mn+m+t} - \frac{L_{k,m}L_{k,t} - \Delta F_{k,m}F_{k,t}}{2} + \frac{L_{k,m}L_{k,mn+m+t} - \Delta F_{k,m}F_{k,mn+m+t}}{2} \right] \end{aligned}$$

Using Lemma(2.6)

$$\sum_{j=0}^{j=n} L_{k,mj+t} = \frac{L_{k,t} - L_{k,mn+m+t} + (-1)^m(L_{k,mn+t} - L_{k,t-m})}{1 + (-1)^m - L_{k,m}}$$

On the other hand, by (23) and (24) we get

$$\begin{aligned} \sum_{j=0}^{j=n} F_{k,mj+t} &= \frac{1}{D} \left[\left(1 - \frac{L_{k,m}}{2}\right)(F_{k,t} - F_{k,mn+m+t}) + \frac{F_{k,m}\Delta}{2}(L_{k,t} - L_{k,mn+m+t}) \right] \\ &= \frac{1}{D} \left[F_{k,t} - F_{k,mn+m+t} - \frac{L_{k,m}F_{k,t} - F_{k,m}L_{k,t}}{2} + \frac{L_{k,m}F_{k,mn+m+t} - F_{k,m}L_{k,mn+m+t}}{2} \right] \end{aligned}$$

Using Lemma(2.6) it follows that

$$\sum_{j=0}^{j=n} F_{k,mj+t} = \frac{F_{k,t} - F_{k,mn+m+t} + (-1)^m(F_{k,mn+t} - F_{k,t-m})}{1 + (-1)^m - L_{k,m}}$$

□

Theorem 3.2 *Let $n \in N$ and $m, t \in Z$, Then*

$$\sum_{j=0}^{j=n} (-1)^j L_{k,mj+t} = \frac{L_{k,t} - L_{k,mn+m+t} + (-1)^m(L_{k,mn+t} - L_{k,t-m})}{1 + (-1)^m - L_{k,m}} \quad (26)$$

and

$$\sum_{j=0}^{j=n} (-1)^j F_{k,mj+t} = \frac{F_{k,t} - F_{k,mn+m+t} + (-1)^m(F_{k,mn+t} - F_{k,t-m})}{1 + (-1)^m - L_{k,m}} \quad (27)$$

Proof. We prove the theorem in two cases by taking n as an even and odd natural number.

Case:1 If n is an even natural number

$$I + (S^m)^{n+1} = (I + S^m) \sum_{j=0}^{j=n} (S^m)^j$$

Since, $\det(I + S^m) = 1 + (-1)^m + L_{k,m} \neq 0$, then using Lemma2.6 we can write

$$I + (S^m)^{-1} (I + (S^m)^{n+1}) S^t = \sum_{j=0}^{j=n} (-1)^j (S^m)^{j+t} = \begin{bmatrix} \frac{1}{2} \sum_{j=0}^{j=n} (-1)^j (L_{k,mj+t}) & \frac{\Delta}{2} \sum_{j=0}^{j=n} (-1)^j (F_{k,mj+t}) \\ \frac{1}{2} \sum_{j=0}^{j=n} (-1)^j (F_{k,mj+t}) & \frac{1}{2} \sum_{j=0}^{j=n} (-1)^j (L_{k,mj+t}) \end{bmatrix} \quad (28)$$

If we take $d = 1 + (-1)^m + L_{k,m}$, then we get

$$\begin{aligned} (I + S^m)^{-1} &= \frac{1}{d} \begin{bmatrix} 1 + \frac{(L_{k,m})}{2} & -\Delta \frac{(F_{k,m})}{2} \\ \frac{(-F_{k,m})}{2} & 1 + \frac{(L_{k,m})}{2} \end{bmatrix} \\ &= \frac{1}{d} \left[\left(1 + \frac{(L_{k,m})}{2}\right) I - \frac{(F_{k,m})}{2} R \right] \end{aligned}$$

Thus it is seen that,

$$(I + S^m)^{-1} (S^t + S^{mn+m+t}) = \frac{1}{d} \left[\left(1 + \frac{(L_{k,m})}{2}\right) I - \frac{(F_{k,m})}{2} R \right] (S^t + S^{mn+m+t})$$

By Corollary (2.2) and $R = S + S^{-1}$ we get

$$R(S^t + S^{mn+m+t}) = \begin{bmatrix} \Delta \frac{(F_{k,t} + F_{k,mn+m+t})}{2} & \Delta \frac{(L_{k,t} + L_{k,mn+m+t})}{2} \\ \frac{(L_{k,t} + L_{k,mn+m+t})}{2} & \Delta \frac{(F_{k,t} + F_{k,mn+m+t})}{2} \end{bmatrix} \quad (29)$$

Using (28) in (27), we obtain

$$\begin{aligned} (I + S^m)^{-1} (S^t + S^{mn+m+t}) &= \frac{1}{d} \left(1 + \frac{L_{k,m}}{2}\right) \begin{bmatrix} \Delta \frac{(L_{k,t} + L_{k,mn+m+t})}{2} & \Delta \frac{(F_{k,t} + F_{k,mn+m+t})}{2} \\ \frac{(F_{k,t} + F_{k,mn+m+t})}{2} & \frac{(L_{k,t} + L_{k,mn+m+t})}{2} \end{bmatrix} \\ &\quad - \frac{F_{k,m}}{2d} \begin{bmatrix} \Delta \frac{(F_{k,t} + F_{k,mn+m+t})}{2} & \Delta \frac{(L_{k,t} + L_{k,mn+m+t})}{2} \\ \frac{(L_{k,t} + L_{k,mn+m+t})}{2} & \Delta \frac{(F_{k,t} + F_{k,mn+m+t})}{2} \end{bmatrix} \end{aligned}$$

Thus from (27) it follows that

$$\begin{aligned} \sum_{j=0}^{j=n} (-1)^j L_{k,mj+t} &= \frac{1}{d} \left[\left(1 + \frac{L_{k,m}}{2}\right) (L_{k,t} + L_{k,mn+m+t}) - \frac{F_{k,m} \Delta}{2} (F_{k,t} + F_{k,mn+m+t}) \right] \\ &= \frac{1}{d} \left[L_{k,t} + L_{k,mn+m+t} + \frac{L_{k,m} L_{k,t} - \Delta F_{k,m} F_{k,t} + (L_{k,m} L_{k,mn+m+t} - \Delta F_{k,m} F_{k,mn+m+t})}{2} \right] \end{aligned}$$

Using Lemma(2.6) and fact that $d = 1 + (-1)^m + L_{k,m}$ we obtain

$$\sum_{j=0}^{j=n} (-1)^j L_{k,mj+t} = \frac{L_{k,t} + L_{k,mn+m+t} + (-1)^m (L_{k,mn+t} + L_{k,t-m})}{1 + (-1)^m + L_{k,m}}$$

Similarly it can be easily seen that by using Lemma(2.6)

$$\sum_{j=0}^{j=n} (-1)^j F_{k,mj+t} = \frac{F_{k,t} + F_{k,mn+m+t} + (-1)^m (F_{k,mn+t} + F_{k,t-m})}{1 + (-1)^m + L_{k,m}}$$

Case:2 If n is an odd natural number

Since n is an odd natural number, we get

$$\sum_{j=0}^{j=n} (-1)^j L_{k,mj+t} = \sum_{j=0}^{j=n-1} (-1)^j L_{k,mj+t} - L_{k,mn+t} \quad (30)$$

Since n is an odd natural number then $(n-1)$ is an even. Then taking $(n-1)$ in (25) and using it in (29), it follows that,

$$\begin{aligned} \sum_{j=0}^{j=n} (-1)^j L_{k,mj+t} &= \frac{L_{k,t} + L_{k,mn+t} + (-1)^m (L_{k,mn-m+t} + L_{k,t-m})}{1 + (-1)^m + L_{k,m}} - L_{k,mn+t} \\ &= \frac{L_{k,t} + (-1)^m (L_{k,mn-m+t} + L_{k,t-m}) - (-1)^m L_{k,mn+t} - L_{k,m} L_{k,mn+t}}{1 + (-1)^m + L_{k,m}} \end{aligned}$$

Using Lemma(2.7) we obtain

$$\sum_{j=0}^{j=n} (-1)^j L_{k,mj+t} = \frac{L_{k,t} + L_{k,mn-m+t} + (-1)^m (L_{k,t-m} - L_{k,mn+t})}{1 + (-1)^m + L_{k,m}} \quad (31)$$

In similar way it can be seen that,

$$\sum_{j=0}^{j=n} (-1)^j F_{k,mj+t} = \sum_{j=0}^{j=n-1} (-1)^j F_{k,mj+t} - F_{k,mn+t} \quad (32)$$

Using (25) in (32), it follows that,

$$\begin{aligned} \sum_{j=0}^{j=n} (-1)^j F_{k,mj+t} &= \frac{F_{k,t} + F_{k,mn+t} + (-1)^m (F_{k,mn-m+t} + L_{k,t-m})}{1 + (-1)^m + L_{k,m}} - F_{k,mn+t} \\ &= \frac{F_{k,t} + (-1)^m (F_{k,mn-m+t} + F_{k,t-m}) - (-1)^m F_{k,mn+t} - L_{k,m} L_{k,mn+t}}{1 + (-1)^m + L_{k,m}} \end{aligned}$$

Using Lemma(2.7) we get expected formula

$$\sum_{j=0}^{j=n} (-1)^j F_{k,mj+t} = \frac{F_{k,t} - L_{k,mn+m+t} + (-1)^m (F_{k,t-m} - F_{k,mn+t})}{1 + (-1)^m + L_{k,m}} \quad (33)$$

□

4 Conclusion:

Some new summation identities have been obtained for the k -Fibonacci and k -Lucas sequences.

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(Concerned with sequences [A000045](#),[A041041](#))
